UDC 532.59

METHOD OF AVERAGING AND THE GENERALIZED VARIATIONAL PRINCIPLE FOR NONSINUSOIDAL WAVES

PMM Vol.36, №1, 1972, pp.71-78

L. A. OSTROVSKII and E. N. PELINOVSKII
(Gor'kii)

(Received June 25, 1970)

A method of asymptotic expansion is proposed for wave processes which are close to stationary periodic waves of arbitrary form. It is shown that equations of the first approximation can be written in the form of Lagrange equations of the second kind for averaged Lagrange and Rayleigh functions.

At the present time most analytical results in the theory of nonlinear wave processes are obtained with the aid of approximate methods based on the smallness of one or another parameter in the initial equations or in the boundary (initial) conditions. Wave processes in media with small nonlinearity and strong dispersion, where the solution is close to one or the superposition of several quasiharmonic waves [1 - 4] have been studied with relatively great completeness. Furthermore, it is well known that in many problems related to waves on the surface of a liquid, in the plasma [5], in transmission lines for electromagnetic waves [6], and also in problems of nonlinear field theory [7] the necessity arises to examine essentially nonsinusoidal waves with arbitrary relationship of nonlinearity and dispersion parameters. Here also some methods exist for obtaining approximate solutions based on local similarity of the process to a stationary traveling wave [5, 8, 9]. From the point of view of generality and physical clearness, apparently, the Hamilton's variational principle in the averaged form as proposed by Whitham [8] is of greatest interest among these methods. In this connection the equations for envelopes (of amplitude, frequency, etc.) of the quasistationary wave are obtained in the form of differential equations of Euler in the corresponding variational problem with averaged Lagrangian. Such an approach, however, is directly applicable only to strictly conservative systems for which the Lagrangian is known (the determination of the latter frequently represents quite a complicated problem [10]). Furthermore, the independent value has the structure of the scheme for asymptotic expansion which permits to obtain averaged equations in any approximation with respect to the small parameter, as it was already done for one nonlinear second order equation [11] and an arbitrary system of first order equations [12].

This paper examines processes which can be described by partial differential equations of the Lagrangian type, including in particular the Rayleigh dissipation function. The asymptotic method permits to examine processes which are locally similar to a plane stationary wave. We succeed in showing that the equations of the first approximation can be derived from the generalized Hamilton's variational principle in the averaged form.

Let us examine the system of equations

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial u_t} + \nabla \frac{\partial L}{\partial \nabla u} - \frac{\partial L}{\partial u} = \Phi, \qquad \Phi = \Phi^{(0)} + \varepsilon\Phi^{(1)} + O(\varepsilon^2) \tag{1}$$

where $u(\mathbf{r}, t)$ is an N-dimensional vector-function, L is the Lagrangian (density of the Lagrange function), Φ is the density of nonpotential forces and ε is the small parameter.

It will be assumed that L and Φ are functions of u, u_t and ∇u and also of "slow" times $\tau = \varepsilon t$ and the coordinate $\rho = \varepsilon\mathbf{r}$. With respect to L and Φ we assume only that these are sufficiently smooth functions of their arguments.

It is known that system (1) can be obtained from the generalized Hamilton's variational principle [13]

$$\int (\delta L + \delta W)\, d\mathbf{r}\, dt = 0, \qquad \delta W = \Phi\delta u \tag{2}$$

We note that for a nonconservative system in the general case it is not permissible to formulate the variational problem corresponding to (2), because a functional does not exist for which the variation coincides with $\delta L + \delta W$ [13].

Together with (1) let us examine the generating system of equations

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial u_t} + \nabla \frac{\partial L}{\partial \nabla u} - \frac{\partial L}{\partial u} = \Phi^{(0)} \qquad (\tau, \rho = \text{const}) \tag{3}$$

It will be assumed that the order of system (1) with respect to $\mathbf{r}$ and t coincides with the order of the generating system (3).

If in the transition from (3) to (1) the order of the system increases then the results obtained below are applicable only to particular classes of stationary waves ("slow" motions in the phase space of system (3)). An example of averaging over "fast" stationary waves is examined in [14].

Let us assume that (3) has solutions in the form of stationary plane waves of the form $u = U(\theta)$, $\theta = \omega t - \mathbf{kr} + \theta_0$, which are determined from the following system of ordinary differential equations

$$\frac{d}{d\theta}\frac{\partial L}{\partial U_\theta} - \frac{\partial L}{\partial U} = \Phi^{(0)} \qquad (\tau, \rho = \text{const}) \tag{4}$$

and depend on two arbitrary constants of integration θ_0 and A and on parameters τ, ρ, ω and $\mathbf{k}$. The frequency ω and the wavenumber $\mathbf{k}$ are connected with A by the dispersion relation $\omega = \omega(\mathbf{k}, A)$ and are selected in such a manner that U is a periodic function of θ with a period 2π.

For given $U(\theta)$, closed trajectories in the phase space of system (4) correspond to periodic solutions. If the generating system is conservative ($\Phi^{(0)} \equiv 0$), then the trajectories occupy some subspace in the phase space. For a nonconservative system the periodic solution U, if it exists, is represented by an isolated trajectory in the phase space; for this solution all A are fixed. The profile of the wave is determined by the actual form of (3) and can differ strongly from the sinusoidal. The determination of conditions for the existence of periodic solutions for an arbitrary system of equations of the form of (4) is in itself a complex problem which has been solved only in some particular cases [15, 16].

The solution of the initial system of equations (1), close to $U(\theta)$, will be sought in the form of asymptotic series with respect to the small parameter ε

$$u = U(\theta, A, \tau, \rho) + \sum_{n=1}^{\infty} \varepsilon^n u^{(n)}(\theta, \tau, \rho)$$

$$\omega(\tau, \rho) = \theta_t, \qquad \mathbf{k}(\tau, \rho) = -\nabla\theta$$

$$A(\tau, \rho) = \sum_{n=0}^{\infty} \varepsilon^n A^{(n)}(\tau, \rho), \qquad \theta = \theta^{(0)}(t, \mathbf{r}, \tau, \rho) + \sum_{n=1}^{\infty} \varepsilon^n \theta^{(n)}(\tau, \rho) \tag{5}$$

Let us expand L and Φ in series with respect to ε taking into consideration (5)

$$\frac{\partial L}{\partial u} = \frac{\partial L}{\partial U} + \varepsilon\left\{\frac{\partial^2 L}{\partial U^2} u^{(1)} + \frac{\partial^2 L}{\partial U \partial U_t}(U_\tau + u_t^{(1)}) + \right.$$
$$\left. + \frac{\partial^2 L}{\partial U \partial \nabla U}(\nabla_\rho U + \nabla u^{(1)})\right\} + O(\varepsilon^2)$$

$$\frac{\partial L}{\partial u_t} = \frac{\partial L}{\partial U_t} + \varepsilon\left\{\frac{\partial^2 L}{\partial U_t \partial U} u^{(1)} + \frac{\partial^2 L}{\partial U_t^2}(U_\tau + u_t^{(1)}) + \right.$$
$$\left. + \frac{\partial^2 L}{\partial U_t \partial \nabla U}(\nabla_\rho U + \nabla u^{(1)})\right\} + O(\varepsilon^2)$$

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial u_t} = \frac{\partial}{\partial t}\frac{\partial L}{\partial U_t} + \varepsilon\left\{\frac{\partial}{\partial \tau}\frac{\partial L}{\partial U_t} + \omega\frac{\partial}{\partial \theta}\left[\frac{\partial^2 L}{\partial U_t \partial U} u^{(1)} + \frac{\partial^2 L}{\partial U_t^2}(U_\tau + u_t^{(1)}) + \right.\right. \tag{6}$$
$$\left.\left. + \frac{\partial^2 L}{\partial U_t \partial \nabla U}(\nabla_\rho U + \nabla u^{(1)})\right]\right\} + O(\varepsilon^2)$$

$$\Phi = \Phi^{(0)} + \varepsilon\left\{\Phi^{(1)} + \frac{\partial \Phi^{(0)}}{\partial U_t}(U_\tau + u_t^{(1)}) + \frac{\partial \Phi^{(0)}}{\partial \nabla U}(\nabla_\rho U + \nabla u^{(1)}) + \frac{\partial \Phi^{(0)}}{\partial U} u^{(1)}\right\} +$$
$$+ O(\varepsilon^2)$$

Analogous expansions can be written for functions $\partial L / \partial \nabla u$ and $\nabla(\partial L / \partial \nabla u)$.

In the right sides of Eqs. (6) the Lagrangian is determined by the principal term of series (5), i.e.

$$L = L[\tau, \rho, U, \omega U_\theta, -\mathbf{k} U_\theta], \ \nabla_\rho U = \partial U / \partial \rho$$

The variables θ and τ, ρ are regarded as independent. Terms of the type $(\partial^2 L / \partial U^2) U$ denote the multiplication of matrix $\partial^2 L / \partial U^2$ by the column U.

Substituting (5) and (6) into (1), equating coefficients for the same powers of ε, and taking into account (4), we obtain

$$T\left(\tau, \rho, \theta, \frac{\partial}{\partial \theta}, \frac{\partial^2}{\partial \theta^2}\right) u^{(n)} = H^{(n)}(\tau, \rho, \theta) \qquad (n = 1, 2, \ldots) \tag{7}$$

$$T u^{(n)} = \frac{\partial}{\partial \theta}\left[\frac{\partial^2 L}{\partial U_\theta^2} u_\theta^{(n)} + \frac{\partial^2 L}{\partial U_\theta \partial U} u^{(n)}\right] - \frac{\partial^2 L}{\partial U^2} u^{(n)} - \frac{\partial^2 L}{\partial U \partial U_\theta} u_\theta^{(n)} -$$
$$- \frac{\partial \Phi^{(0)}}{\partial U} u^{(n)} - \frac{\partial \Phi^{(0)}}{\partial U_\theta} u_\theta^{(n)} \tag{8}$$

$$H^{(1)} = \Phi^{(1)} + \frac{\partial \Phi^{(0)}}{\partial U_t} U_\tau + \frac{\partial \Phi^{(0)}}{\partial \nabla U} \nabla_\rho U + \frac{\partial^2 L}{\partial U \partial U_t} U_\tau +$$
$$+ \frac{\partial^2 L}{\partial U \partial \nabla U} \nabla_\rho U - \frac{\partial}{\partial \tau}\frac{\partial L}{\partial U_t} - \nabla_\rho \frac{\partial L}{\partial \nabla U} - \omega \frac{\partial}{\partial \theta}\left[\frac{\partial^2 L}{\partial U_t^2} U_\tau + \frac{\partial^2 L}{\partial U_t \partial \nabla U} \nabla_\rho U\right] +$$
$$+ \mathbf{k}\frac{\partial}{\partial \theta}\left[\frac{\partial^2 L}{\partial \nabla U \partial U_t} U_\tau + \frac{\partial^2 L}{\partial (\nabla U)^2} \nabla_\rho U\right] \tag{9}$$

Expressions for $H^{(n)}$ at $n \geqslant 2$ are cumbersome and are not presented here.

Thus, the determination of functions A, θ and $u^{(n)}$ in any approximation is connected with finding the solution of linear system (7) with periodic coefficients and a right side which is periodic with respect to θ. This solution can be represented in the form

$$u^{(n)} = YC^{(n)} + Y\int_0^\theta Y^* H^{(n)}\, d\theta' \tag{10}$$

Here $C^{(n)}$ is a constant vector, Y is a matrix composed of vectors of the fundamental system of solutions of equations in variations of $T\psi = 0$, $Y^*$is the analogous matrix for the conjugate system which is connected with Y by the relationships

$$\frac{dY}{d\theta} Y^* = \left[\frac{\partial^2 L}{\partial U_\theta^2}\right]^{-1}, \qquad YY^* = 0 \tag{11}$$

Two particular solutions of equations in variations are directly determined in terms of U [12, 16]

$$Y_1 = U_\theta, \qquad Y_2 = v + \alpha\theta U_\theta \tag{12}$$

where U_θ and v are periodic functions of θ, α is a constant determined by the dependence of ω and k on A. By virtue of the theorem of Floquet the remaining $2N-2$ vectors of matrix Y can be written in the form

$$Y_i = e^{\lambda_i \theta} f_i(\theta) + \text{compl. conj.} \qquad (i = 3, 4, \ldots, 2N) \tag{13}$$

in this case $f(\theta + 2\pi) \equiv f(\theta)$ and all λ_i are considered to be distinct. It is assumed that if for any characteristic exponent λ_l $\operatorname{Re}\lambda_l = 0$, then $\operatorname{Im}\lambda_l \neq \pm n$ $(n = 0, 1, 2, \ldots)$. This condition is commonly used in the theory of quasi-linear oscillations as the condition for the absence of internal resonance [17]. According to (11) - (13), the matrices Y and Y^* have the form [18]

$$\begin{aligned} Y_{ij} &= y_{ij} e^{\lambda_j \theta} + \alpha\theta y_{i1}\delta_{j2} + \text{compl. conj.} \\ Y_{jk}^* &= y_{jk}^* e^{-\lambda_j\theta} - \alpha\theta y_{2k}^* \delta_{j1} + \text{compl. conj.} \end{aligned} \tag{14}$$

where y_{ij} and y_{ik}^* are periodic functions of θ and $\lambda_1 = \lambda_2 = 0$.

Substituting Y and Y^* into (10), we obtain after transformations

$$\begin{aligned} u_i^{(n)} = {} & y_{ij} e^{\lambda_j\theta} C_j^{(n)} + \alpha\theta y_{i1} C_2^{(n)} + y_{ij} e^{\lambda_j\theta} \int_0^\theta y_{jk}^* e^{-\lambda_j\theta'} H_k^{(n)}\, d\theta' + \\ & + \alpha y_{i1} \int_0^\theta d\theta' \int_0^{\theta'} y_{2k}^* H_k^{(n)}\, d\theta'' + \text{compl. conj.} \end{aligned} \tag{15}$$

It follows from this that $u^{(n)}$ is a bounded function of θ on satisfying the following necessary conditions:

$$C_j^{(n)} = \begin{cases} -\int_0^\infty y_{jk}^* H_k^{(n)} e^{-\lambda_j\theta} d\theta, & \operatorname{Re}\lambda_j > 0 \\ \int_{-\infty}^0 y_{jk}^* H_k^{(n)} e^{-\lambda_j\theta} d\theta, & \operatorname{Re}\lambda_j < 0 \end{cases} \tag{16}$$

$$2\pi\alpha C_2^{(n)} + \int_0^{2\pi} y_{1k}^* H_k^{(n)}\, d\theta + \alpha \int_0^{2\pi} d\theta \int_0^\theta y_{2k}^* H_k^{(n)}\, d\theta' = 0 \tag{17}$$

$$\int_0^{2\pi} y_{2k}^* H_k^{(n)}\, d\theta = 0 \tag{18}$$

Consequently there is only one independent equation for two unknown functions $A^{(n)}$

and $\theta^{(n)}$. Such a situation is characteristic for nonisochronal oscillating systems and is connected with the fact that for such systems the amplitude and the phase are found from equations of different approximations [17]. For finding $A^{(n)}$ and $\theta^{(n)}$ in some cases it is convenient to utilize the following equations together with (18):

$$\int_0^{2\pi} y_{1k}^* H_k^{(n)} d\theta = 0 \tag{19}$$

in order to be able to make the limit transition to a quasi-harmonic wave. In fact, in the quasi-linear case ($\partial\omega / \partial A = 0$) in (17) $\alpha = 0$, and condition (19) becomes necessary. In this connection there is no ambiguity in the selection of equations for $A^{(n)}$ and $\theta^{(n)}$.

Equations (15) - (19) permit to find sequentially the unknowns $A^{(n)}$, $\theta^{(n)}$ and $u^{(n)}$. It should be noted that series (5) correspond to a particular class of boundary conditions, namely to the representation of a wave propagating only in one direction. An analogous situation occurs also for linear partial differential equations which contain the small parameter [19].

Since θ does not enter explicitly into (18) we can regard $A^{(n)}$, $\omega^{(n)}$, $\mathbf{k}^{(n)}$ as unknown functions. In this case it is necessary to supplement (18) by the following equations:

$$\partial\mathbf{k}/\partial t + \nabla\omega = 0, \qquad \operatorname{rot}\mathbf{k} = 0 \tag{20}$$

Let us examine more closely the equations of the first approximation. It is evident that equations (18) and (20) for $n = 1$ are quasi-linear with respect to A, ω and $\mathbf{k}$ and that they belong either to the hyperbolic or elliptic type (the latter is possible only for nonlinear systems [9]). In the hyperbolic case a family of real characteristics exists which are rays in the $\mathbf{r}$, t space. In this sense the present method can be regarded as a generalization of the space-time geometrical optics to nonlinear media.

It is emphasized that the obtained equations are valid for systems with arbitrary (not necessarily small) nonlinearity. The results of some papers [9, 20, 21], which are presented for the case of small nonlinearity without concrete definition of dispersion parameters and consequently the form of the stationary wave, are actually valid only for strong dispersion when the stationary wave is close to the harmonic wave.

We can show that in the absence of nonconservative forces $\Phi^{(0)}$ the operator T is self-conjugate. This is also valid for $\Phi^{(0)} \neq 0$, if $\Phi^{(0)}$ satisfies the conditions

$$\frac{\partial\Phi_i^{(0)}}{\partial U_{j\theta}} \equiv -\frac{\partial\Phi_j^{(0)}}{\partial U_{i\theta}}, \qquad \frac{\partial\Phi_i^{(0)}}{\partial U_j} \equiv \frac{\partial\Phi_j^{(0)}}{\partial U_i} - \frac{d}{d\theta}\frac{\partial\Phi_j^{(0)}}{\partial U_{i\theta}}$$

For functions y_{1k}^* and y_{2k}^* in this case we can take $U_{A'}$ and U_θ, respectively.

It will be shown that the equation of the first approximation can be written in the Lagrangian form. For this purpose we substitute (9) into (18)

$$\int_0^{\pi} U_\theta\left\{\omega\frac{\partial}{\partial\theta}\left[\frac{\partial^2 L}{\partial U_t^2}U_\tau + \frac{\partial^2 L}{\partial U_t\partial\nabla U}\nabla_\rho U\right] - \mathbf{k}\frac{\partial}{\partial\theta}\left[\frac{\partial^2 L}{\partial\nabla U\,\partial U_t}U_\tau + \frac{\partial^2 L}{\partial(\nabla U)^2}\nabla_\rho U\right] + \right.$$

$$\left. + \frac{\partial^2 L}{\partial U\,\partial U_t}U_\tau + \frac{\partial^2 L}{\partial U\partial\nabla U}\nabla_\rho U + \frac{\partial}{\partial\tau}\frac{\partial L}{\partial U_t} + \nabla_\rho\frac{\partial L}{\partial\nabla U}\right\} d\theta = \tag{21}$$

$$= \int_0^{2\pi} U_\theta\left\{\Phi^{(1)} + \frac{\partial\Phi^{(0)}}{\partial U_t}U_\tau + \frac{\partial\Phi^{(0)}}{\partial\nabla U}\nabla_\rho U\right\} d\theta$$

Let us integrate by parts the integral containing the brackets, taking into account the periodicity of all functions with respect to θ. Then, after simple transformations the left side of (21) can be written in the form

$$\int_0^{2\pi}\left\{\frac{\partial}{\partial\tau}\left(\frac{\partial L}{\partial U_t}U_\theta\right)+\nabla_\rho\left(\frac{\partial L}{\partial\nabla U}U_\theta\right)\right\}d\theta$$

In this manner (18) can be presented in the form

$$\frac{\partial}{\partial\tau}\frac{\partial\langle L\rangle}{\partial\omega}-\nabla_\rho\frac{\partial\langle L\rangle}{\partial\mathbf{k}}=\frac{\langle\delta W\rangle}{\partial\theta} \tag{22}$$

Dispersion relation can be written in the form

$$\frac{\partial\langle L\rangle}{\partial A}=-\frac{\langle\delta W\rangle}{\delta A} \tag{23}$$

Here $\langle L\rangle$ is the value of the Lagrangian averaged over the period, $\langle\delta W\rangle$ is the average virtual work. The symbols $\langle\delta W\rangle/\delta\theta$ and $\langle\delta W\rangle/\delta A$ denote coefficients in the variations $\delta\theta$ and δA in the expression

$$\langle\delta W\rangle=\langle\Phi U_\theta\rangle\,\delta\theta+\langle\Phi U_A\rangle\,\delta A \tag{24}$$

In this manner we arrive in the first approximation at Lagrange equations of the second kind for a certain system described through generalized functions A and θ. Consequently we can formulate the generalized Hamilton's variational principle in the averaged form

$$\int(\delta\langle L\rangle+\langle\delta W\rangle)\,d\mathbf{r}dt=0 \tag{25}$$

From this the method of construction of equations of the first approximation is clear: the Lagrangian and the virtual work should be averaged over the period of the wave and the corresponding Lagrange equations of the second kind for the function A and θ should be written. This very approach was proposed in [8] for conservative systems. The method presented above provides some support for this approach.

As in mechanics, nonconservative (we also include here active systems for which the dissipation can be negative or positive) distributed systems can be described in terms of density $\Phi=-\partial R/\partial U_t$ of the Rayleigh function. If average values $\langle R\rangle$ of the Rayleigh function are introduced, then we can show that (22) can be written in the form

$$\frac{\partial}{\partial\tau}\frac{\partial\langle L\rangle}{\partial\omega}-\nabla_\rho\frac{\partial\langle L\rangle}{\partial\mathbf{k}}=-\frac{\partial\langle R\rangle}{\partial\omega} \tag{26}$$

With the aid of (26) we can obtain the transfer equation for the average values of energy density $\langle E\rangle$ and energy flux $\langle S\rangle$ in dissipative media

$$\frac{\partial\langle E\rangle}{\partial\tau}+\mathrm{div}_\rho\langle\mathbf{S}\rangle=-\omega\frac{\partial\langle R\rangle}{\partial\omega}-\frac{\partial\langle L\rangle}{\partial\tau} \tag{27}$$

$$\left(\langle E\rangle=\omega\frac{\partial\langle L\rangle}{\partial\omega}-\langle L\rangle,\qquad\langle\mathbf{S}\rangle=-\omega\frac{\partial\langle L\rangle}{\partial\mathbf{k}}\right)$$

Here $\omega\partial\langle R\rangle/\partial\omega$ is the density of dissipated power and $\partial\langle L\rangle/\partial\tau$ characterizes the energy change connected with the nonstationary behavior of parameters of the medium.

We note here the case where $\partial\langle R\rangle/\partial\omega=\beta\partial\langle L\rangle/\partial\omega$, where β is a constant. Then (26) is written in the form of the Lagrange equation of a "reduced" conservative system

$$\frac{\partial}{\partial\tau}\frac{\partial\langle L^*\rangle}{\partial\omega}-\nabla_\rho\frac{\partial\langle L^*\rangle}{\partial\mathbf{k}}=0\qquad(L^*=Le^{\beta\tau}) \tag{28}$$

In this case we can formulate the variational problem for the functional of "reduced" action $\int L^* drdt$ (and analogously for $\int \langle L^*\rangle\, drdt$). The general form of L^* for dissipative systems with concentrated parameters is discussed in [22, 23].

For a wavepacket localized in space it follows from (28) that

$$\int \frac{\partial \langle L^*\rangle}{\partial \omega}\, d\mathbf{r} = \text{const} \tag{29}$$

and the quantity of the integral in the left side of (29) can be called "reduced" adiabatic invariant in analogy to the adiabatic invariant in the conservative system. For a concentrated system with one degree of freedom a relationship analogous to (29) was obtained in [17].

We note in conclusion that all obtained equations are also directly applicable to oscillating systems with concentrated parameters for any number of degrees of freedom, if we assume $\nabla \equiv 0$ and if L and R are understood to be functions of Lagrange and of the Rayleigh system respectively. Corresponding results are, of course, identical to known results [17]. However here also the approach used above has some advantages because the Lagrangian formulation of the method allows to obtain a number of results in a simpler form, especially for strongly nonlinear oscillations in systems with many degrees of freedom.

BIBLIOGRAPHY

1. Mitropol'skii, Iu. A. and Moseenkov, B. M., Lectures on Application of the Asymptotic Methods to Solution of Partial Differential Equations. Inst. of Mathematics Akad. Nauk UkSSSR, Kiev, 1968.
2. Taniuti, T. and Yajma, N., A perturbation method for nonlinear wave modulation. I. J. Math. Phys. Vol. 10, №8, 1969.
3. Rabinovich, M. I., On the asymptotic method in the theory of nonlinear oscillations of distributed systems. Dokl. Akad. Nauk SSSR Vol. 191, №6, 1970.
4. Gaponov, A. V., Ostrovskii, L. A. and Rabinovich, M. I., One-dimensional waves in nonlinear systems with dispersion. Izv. Vyssh. Uchebn. Zavedenii, Radiofizika Vol. 13, №2, 1970.
5. Whitham, G. B., Nonlinear dispersion waves. Proc. Roy. Soc., Ser. A 283, N 1393, 1965.
6. Filippov, Iu. F., Theory of propagation of stationary waves of finite amplitude in ferrites. Izv. Vyssh. Uchebn. Zavedenii, Radiofizika Vol. 8, №2, 1965.
7. Kurdgelaidze, D. F., Theory of nonlinear field $(\square - \alpha\varphi^2)\varphi = 0$. JETP Vol. 36, №4, 1959.
8. Whitham, G. B., A general approach to linear and nonlinear dispersive waves using a Lagrangian. J. Fluid. Mech. Vol. 22, №2, p. 273, 1965.
9. A discussion on nonlinear theory of wave propagation in dispersive systems (organized by M. J. Lighthill). Proc. Roy. Soc., Ser. A 299, №1456, 1967.
10. Seliger, R. L. and Whitham, G. B., Variational principle in continuum mechanics. Proc. Roy. Soc., Ser. A 305, №1480, 1968.
11. Luke, J. C., A perturbation method for nonlinear dispersive wave problems. Proc. Roy. Soc., Ser. A 292, №1430, 1966.
12. Ostrovskii, L. A. and Pelinovskii, E. N., Method of averaging for nonsinusoidal waves. Dokl. Akad. Nauk SSSR, Vol. 195, №4, 1970.
13. Lur'e, A. I., Analytical Mechanics. M., Fizmatgiz, 1961.

14. Rabinovich, M.I., Method of averaging over stationary waves. Izv. Vyssh. Uchebn. Zavedenii, Radiofizika Vol.10, №2, 1967.
15. Kulikov, N.K., Conditions for the existence and determination of parameters of periodic motions of autonomous systems. Izv. Akad. Nauk SSSR, Mekhanika i Mashinostroenie, №4, 1959.
16. Malkin, I.G., Some Problems in the Theory of Nonlinear Oscillations. M., Gostekhizdat, 1956.
17. Mitropol'skii, Iu. A., Problems of the Asymptotic Theory of Nonstationary Oscillations. 2nd Ed. M., "Nauka", 1964.
18. Gantmakher, F.P., Matrix Theory. 2nd ed., M., "Nauka", 1966.
19. Courant, R., Partial Differential Equations, New York, Interscience, 1962.
20. Karpman, V.I. and Krushkal', E.M., Modulated waves in nonlinear dispersive media. JETP Vol. 55, №2, 1968.
21. Tam, G.K.W., Nonlinear dispersion of cold plasma waves. J. Plasma Phys. Vol. 4, №1, 1970.
22. Vaast, R., Variational principle for certain nonconservative systems. Amer. J. Phys. Vol. 35, №5, p.419, 1967.
23. Kas'ianov, V.A. and Tkachenko, N.E., Description of dissipative systems with the aid of Hamiltonian formalism. Prikl. Mekh. Vol.6, №2, 1970.

Translated by B.D.

UDC 534.12

ON NONLINEAR TRANSVERSE RESONANT OSCILLATIONS IN AN ELASTIC LAYER AND A LAYER OF PERFECTLY CONDUCTING FLUID

PMM Vol. 36, №1, 1972, pp.79-87
N.R. SIBGATULLIN
(Moscow)
(Received April 14, 1971)

One proves the equivalence of the equations of the one-dimensional plane motion of an isotropic nonlinearly-elastic body and that of a perfectly conducting compressible fluid moving in an external magnetic field, the magnetic permeability of the fluid being an arbitrary function of the density and of the modulus of intensity of the magnetic field. For these models of the continuous medium one considers essentially the nonlinear problem of the transverse oscillations induced in an infinite layer by the periodic action of external tangential forces at one of the plane boundaries, while at the other one a perfect reflection of the waves is assumed. The singularity of the bahavior of the forced resonant oscillations are developed in the case when in the elastic body the velocity of the longitudinal waves is much larger than the velocity of the transverse waves and in the fluid the velocity of the sound exceeds by far the velocity of the Alfvén waves. One establishes the relation between the amplitude of the constraining forces and